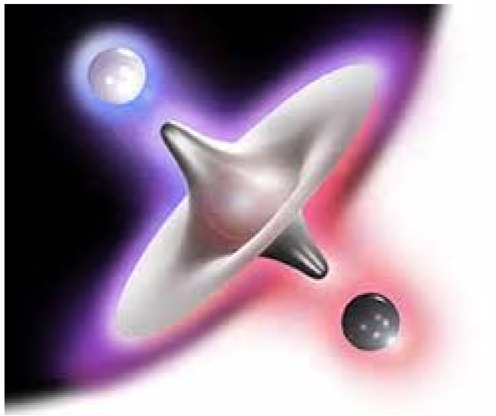
Salon of Science

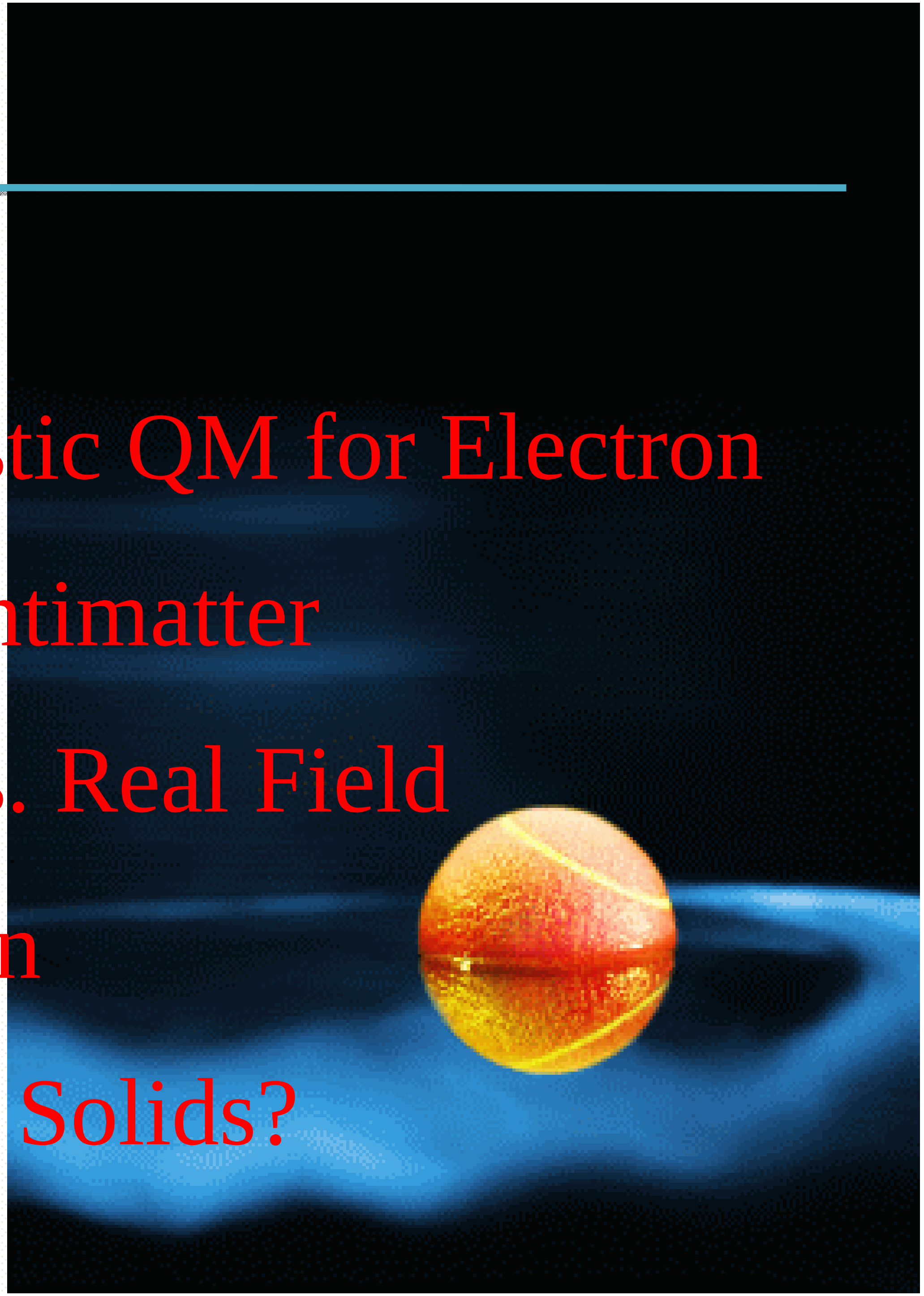
The Conceptual Origin of Majorana Fermion



Cao Zexian

2014.11.01

Outline



- QM and Relativistic QM for Electron
- Antiparticle & Antimatter
- Complex Field vs. Real Field
- Majorana Fermion
- Searching MF, in Solids?

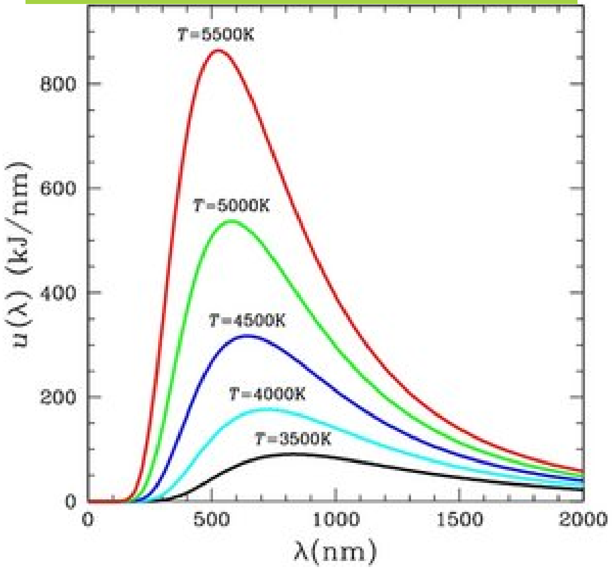
QM begins with the understanding of gas dynamics (Boltzmann 1877) and black-body radiation (Planck 1900)

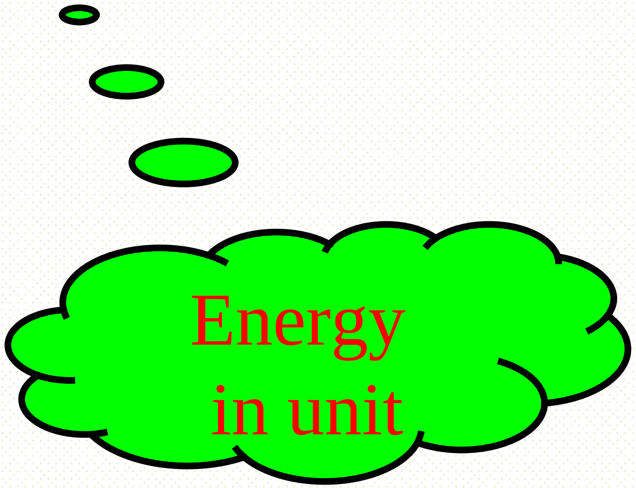
$$n_p \propto \exp(-\beta p)$$

$$\frac{\partial^2 S_\nu}{\partial U_\nu^{\ 2}} = -\frac{k}{U_\nu (h\nu + U_\nu)}$$

$$n_0 + n_1 + n_2 + \ldots + n_p = n$$

$$0 \cdot n_0 + 1 \cdot n_1 + \ldots + p \cdot n_p = E$$





To distribute the energy U_N (P portions) among N oscillators (for the calculation of entropy with classical concept of billiard balls) with frequency υ.

Suppose U_N is indefinitely large, divisible, and the total number P of equal part

Dies diskrettisierung ist für die Argumentation unentbehrlich!

$$P = U/\varepsilon = U/h\nu$$

Planck called it 'Energieelement'

$$W = \frac{(P+N-1)!}{P!(N-1)!}$$

$$S_N = Nk[(1+\tfrac{U}{\varepsilon})\ln(1+\tfrac{U}{\varepsilon}) - \tfrac{U}{\varepsilon}\ln\tfrac{U}{\varepsilon}]$$

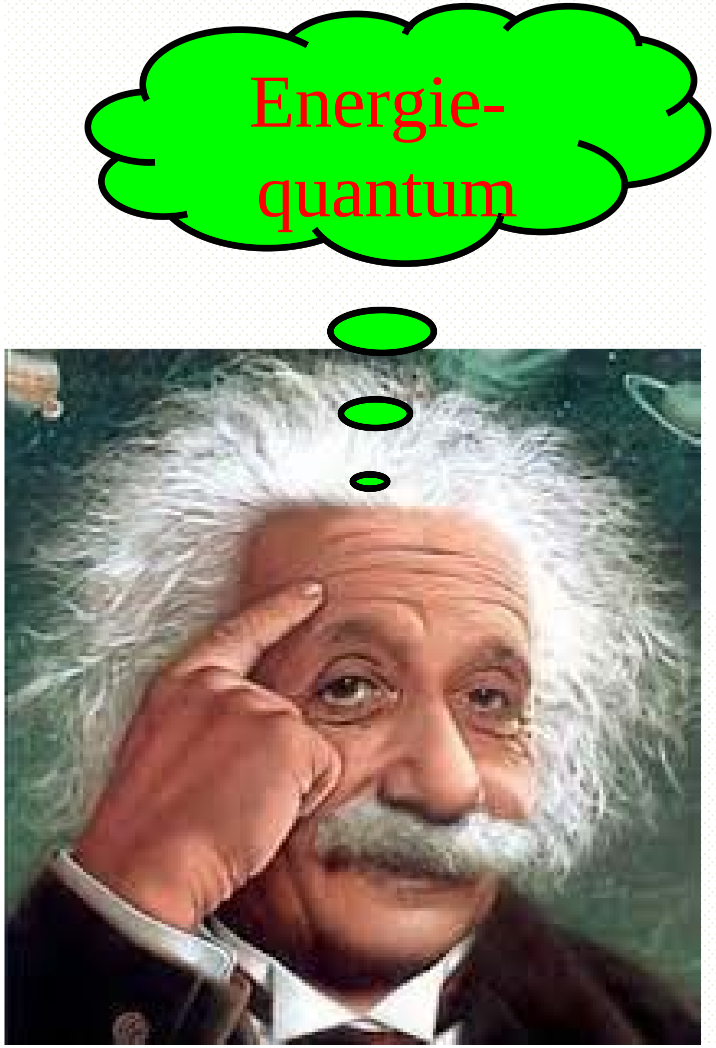
$$\frac{1}{T} = \frac{\partial S}{\partial U}$$

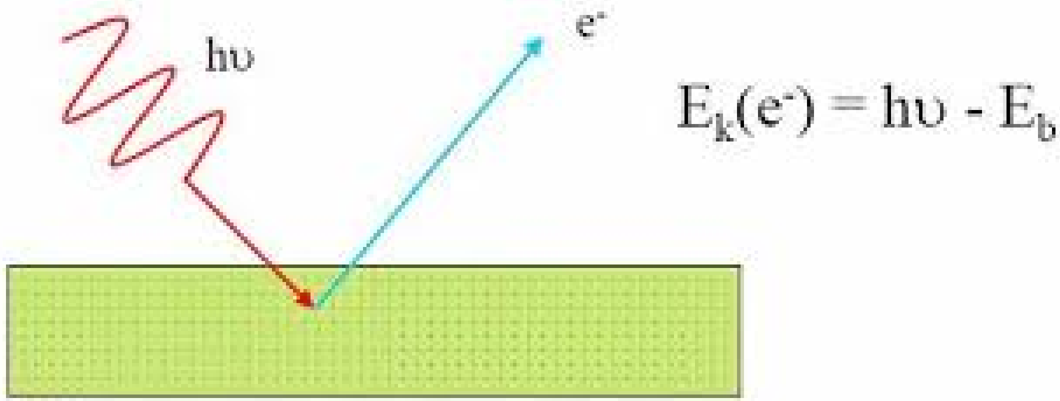
$$U = \frac{\varepsilon}{\exp(\varepsilon/kT)-1}$$

$$e_v = \frac{4\nu^2}{c^2}\frac{h\nu}{\exp(h\nu/kT)-1}$$

M. Planck, Annalen der Physik, 4, 553(1901).

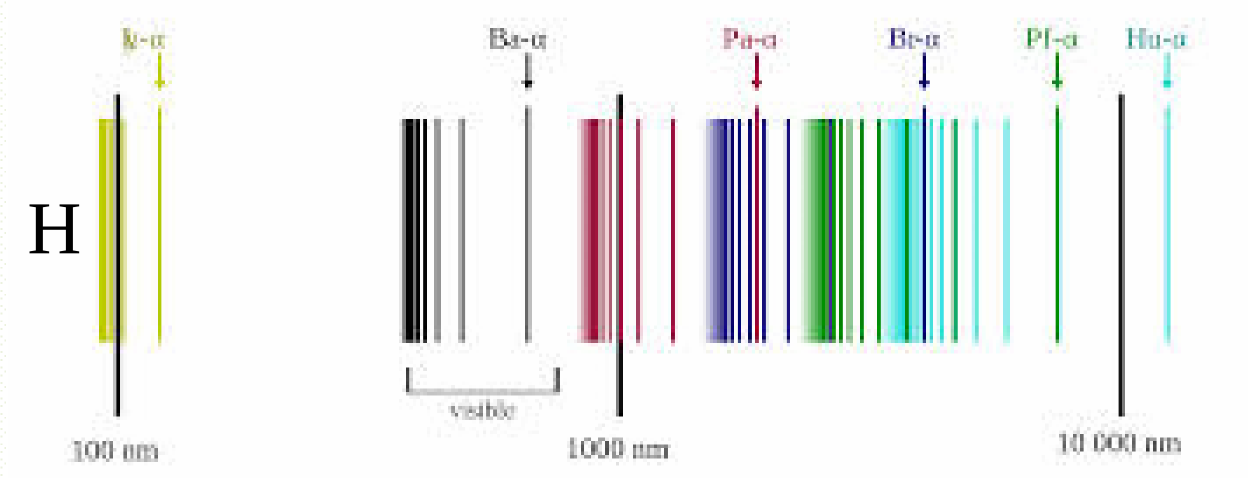
The concept of Energy Quanta was taken by Einstein in 1905 to the interpretation of photoelectron effect

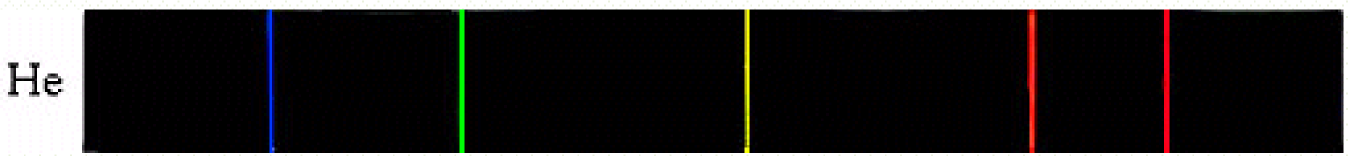




In 1908 Johannes Stark applied it to ionization of gases and to photochemical reactions.

And QM proceeded with the understanding of spectral features of radiation of ELECTRON





Spectral lines arise from the ‘jump’ of ELECTRON between different stationary states. Then came the question: what is ELECTRON, of which the existence was conceived from observation of chemical reactions.

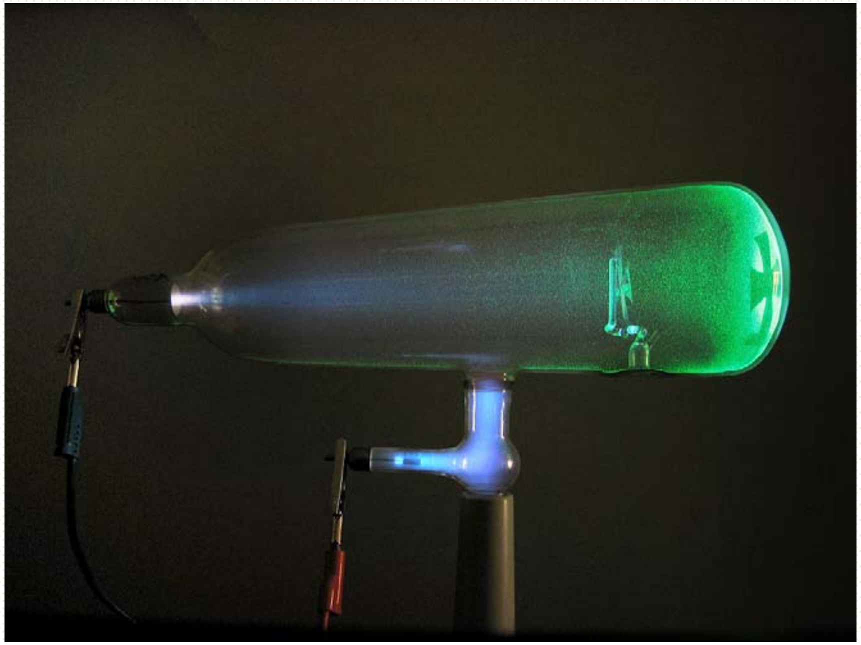
ELECTRON ~cathode radiation,

thus ELECTRON are particles.

Crookes tube

~Maltese Cross

~Projection



王子说：让电子是波！→ Matter wave

and these are the frequency and wavelength of that wave

Louis de Broglie
(1892-1987)

$$\nu = E / h$$

$$\lambda = h / p$$

Max von Laue: Welle? That may need a wave equation. Herr Schrödinger, would you please try to formulate a wave equation for it?

What is wave?

Well, let me see.

$$e^{i(kx-\omega t)}$$

For relativistic Electron

$$E^2 = p^2c^2 + m^2c^4$$

$$\Downarrow$$

$$(\hbar^2\nabla^2 - \mathrm{m}^2\mathrm{c}^2)\psi = 0$$

Schrödinger gave it up since the current is non-conservative.

How Schrödinger got his equation?

$$S = k \log W \longrightarrow \Psi = A \exp(S / i\hbar)$$

Hamilton-Jacobi eq.

$$H + \partial S / \partial t = 0 \quad \Longrightarrow \quad -i\hbar \partial \psi / \partial t = H \psi$$

$$e^{i(kx - \omega t)}$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x_i{}^2} + V \psi = i\hbar \frac{\partial \psi}{\partial t}$$

Quantisierung als Eigenwertproblem

Ervin Schrödinger

Erste Mitteilung: Ann. Phys. 79, 361(1926)

Zweite Mitteilung: Ann. Phys. 79, 489(1926)

Dritte Mitteilung: Ann. Phys. 80, 437(1926)

Vierte Mitteilung: Ann. Phys. 81, 109(1926)

But electron must be relativistic. Dirac, a clever guy armed with engineering mathematics, stepped on the stage.

Heisenberg-Born

$$QP - PQ = i\hbar$$

Dirac, 23 years old, calculated the classical Poisson Bracket

$$[u_1 u_2, v_1 v_2] \mapsto \frac{u_1 v_1 - v_1 u_1}{[u_1, v_1]} = \frac{u_2 v_2 - v_2 u_2}{[u_2, v_2]}$$

$$uv - vu = k[u, v]$$

$$k = i\hbar$$

$$uv - vu = i\hbar[u, v]$$

Quantum Commutation vs. Classical Poisson Bracket

The Fundamental Equations of Quantum Mechanics.

By P. A. M. Dirac, 1851 Exhibition Senior Research Student, St. John's College, Cambridge.

(Communicated by R. H. Fowler, F.R.S.—Received November 7th, 1925.)

§ 1. *Introduction.*

Quantum quantities

We make the fundamental assumption that *the difference between the Heisenberg products of two quantum quantities is equal to $ih/2\pi$ times their Poisson bracket expression.* In symbols,

$$xy - yx = ih/2\pi \,.\, [x, y]. \qquad (11)$$

In the same paper (1926), he also noticed the special **statistics** for electron

$$\psi(1,2) = a_{mn}\varphi_m(1)\varphi_n(2) + b_{mn}\varphi_m(2)\varphi_n(1)$$

$$a_{nm} = b_{nm}$$

Bose-Einstein Statistics (1924-1925)

$$a_{nm} = -b_{nm}$$

Fermi-Dirac Statistics (1926)

Fermi-Dirac Statistics

$$n_i = \frac{1}{e^{(\varepsilon-\mu)/kT} + 1}$$

Suppose there are levels labeled by index i, energy ε_i, degeneracy g_i , with n_i particle on them.

For each sublevel, particle number is 0 or 1.

For each energy level,

$$W = \prod_i w(n_i, g_i) = \prod_i \frac{g_i!}{n_i!(g_i - n_i)!}$$

$$w(n_i, g_i) = \frac{g_i!}{n_i!(g_i - n_i)!}$$

$$f(n_i) = \ln W + \alpha(N - \sum_i n_i) + \beta(E - \sum_i \varepsilon_i n_i)$$

$$n_i = \frac{g_i}{e^{\alpha+\beta\varepsilon_i} + 1}$$

Dirac embarked on Relativistic QM for Electron

Relativity:

$$E^2 = p^2c^2 + m^2c^4$$

$$H \mapsto i\hbar\partial_t \qquad \hat{p}_x \mapsto -i\hbar\partial_x$$

$$\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = m^2c^2$$

K-G equation

自旋为零
的粒子

Relativistic？ Yes. Particle number conservation? No.

Dirac: Relativistic QM for Electron

Relativity: $E^2 = p^2c^2 + m^2c^4$

F-D statistics?

$$x^2 + y^2 = (\alpha x + \beta y)^2$$

$$\alpha^2 = \beta^2 = 1$$
$$\alpha\beta + \beta\alpha = 0$$

$$E = c\vec{\alpha} \cdot \vec{p} + \beta mc^2$$

A proper choice for α , β is 4 by 4 matrices.

Pauli Matrices, Dirac Matrices

For Pauli matrices, Tr σ ＝0, eigenvalues: 1, -1.

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}$$

Complex fields

$$\psi_{z+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \psi_{z-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \psi_z = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

Pauli Matrices,   Dirac Matrices

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}; \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$i\hbar\partial_t \psi = (c\vec{\alpha}\cdot\vec{p} + \beta mc^2)\psi$$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

$$i\gamma\cdot\partial\psi = m\psi$$

$$\gamma^0 = \beta$$

$$\gamma^k = \beta\alpha^k \qquad k = 1,2,3$$

Dirac equation implies intrinsic spin for electron

$$\frac{d\hat{L}}{dt} = \frac{d(\hat{r} \times \hat{p})}{dt} = [\hat{L}, c\vec{\alpha} \cdot \vec{p} + mc^2\beta] / i\hbar$$

$$= c(\vec{\alpha} \times \vec{p})$$

不守恒

$$\hat{L} \mapsto \hat{L} + \hat{S}$$

$$S = \tfrac{\hbar}{2}\Sigma; \quad \Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

哈，守恒啦

$$\frac{d(\hat{L} + \hat{S})}{dt} = 0$$

$$\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

Eigenvalues:1， -1； spin 1 / 2

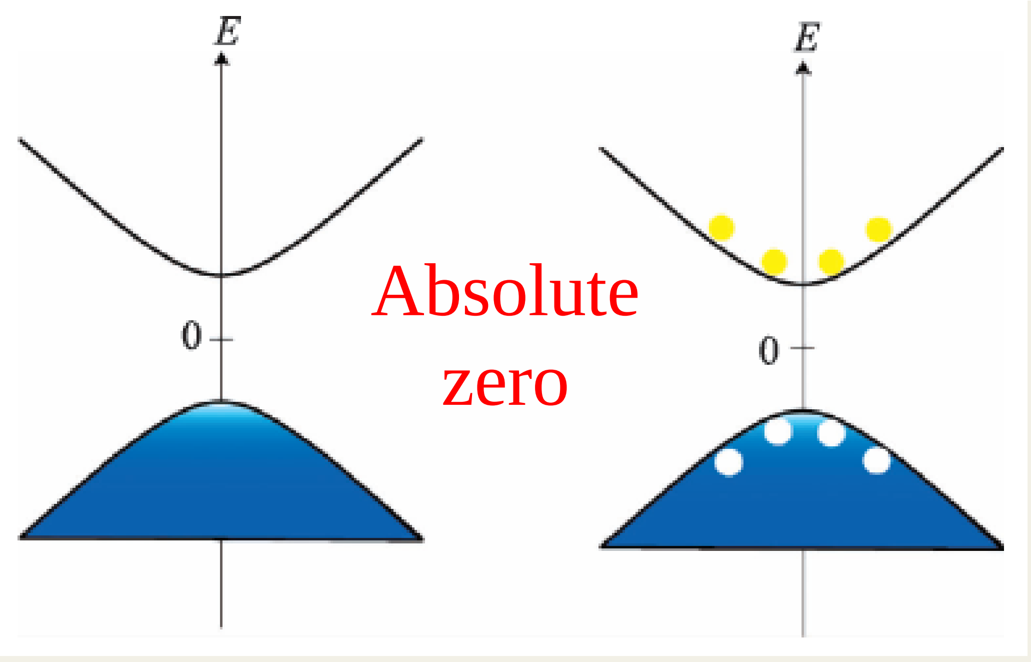
Interpretation of Dirac Equation

$$i\gamma \cdot \partial \psi = m\psi$$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

e^- ; E

e^- ; -E

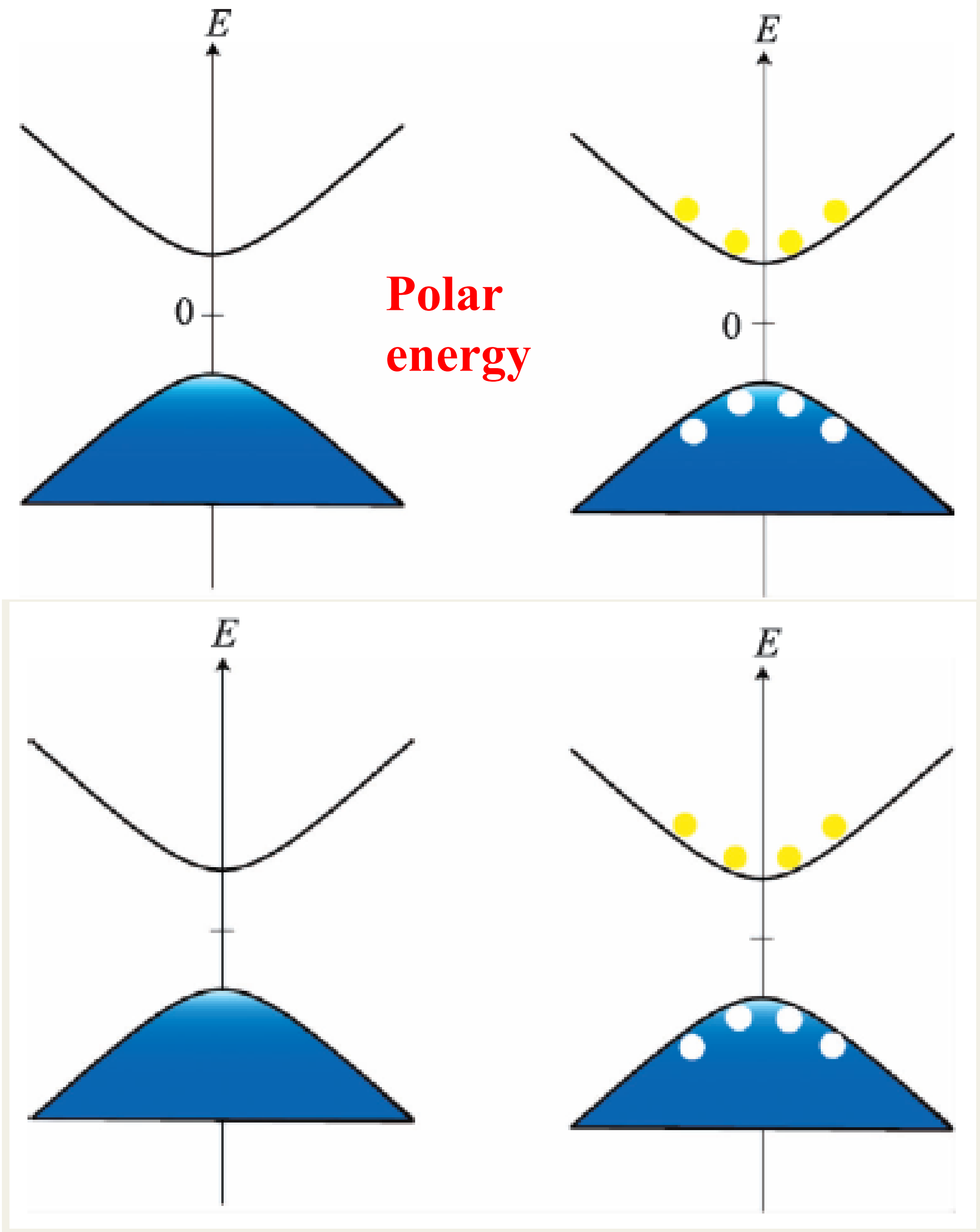


Negative electron sea?

Hole, positive charge?

Proton, or anti-electron?

Nonsensical "Positive Hole" of Dirac



Object of Dirac equation: Electrons only! Why should the "hole" left behind be positively charged?

Object of Schrödinger equation for solid: Electrons on the background of positively charged ions.

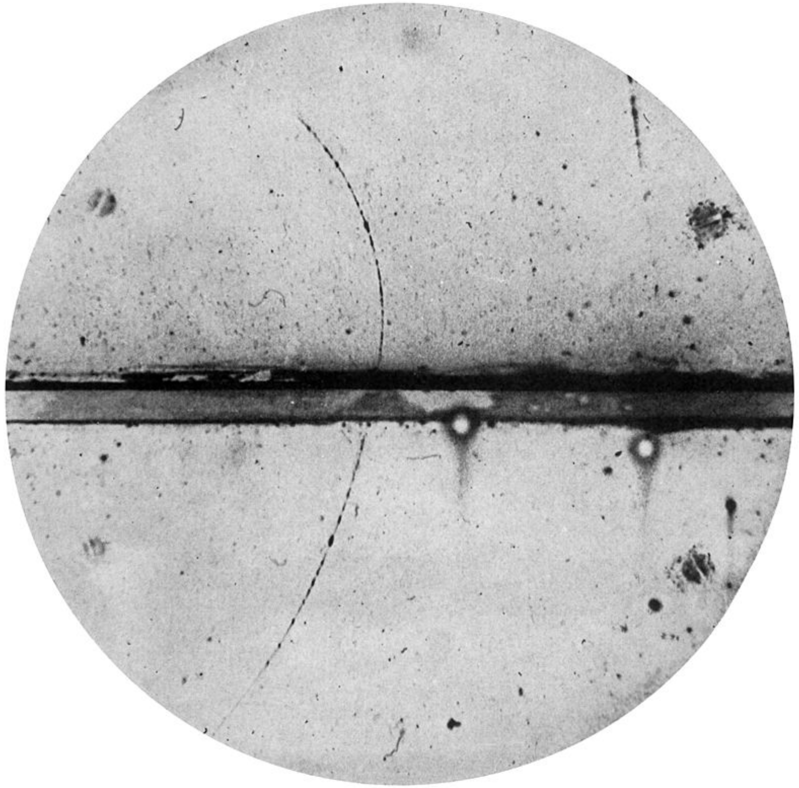
Interpretation of Dirac Equation

$$i\gamma\cdot\partial\psi = m\psi$$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

e^- ; E

e^+ ; E

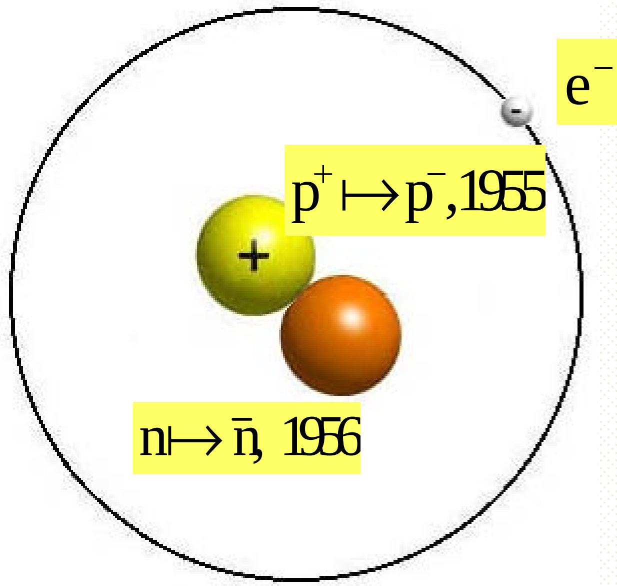


Dmitri Skobeltsyn 1929

Chung-Yao Chao 1929

Carl D. Anderson 1932

More Anti-particles



$$i\hbar\partial_t\psi = (c\vec{\alpha}\cdot\vec{p} + \beta mc^2)\psi$$

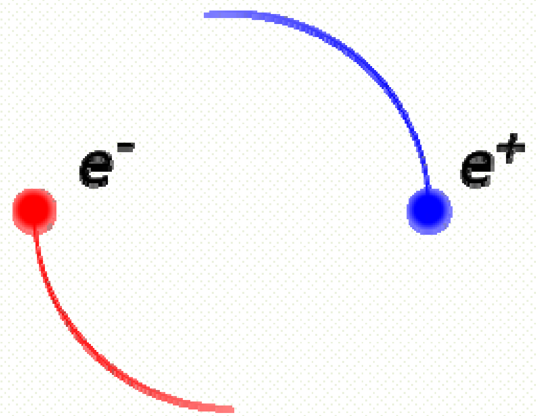
Baryon number: 1 for neutron, -1 for antineutron

Anti-Matter

antimatter is material composed of **antiparticles**

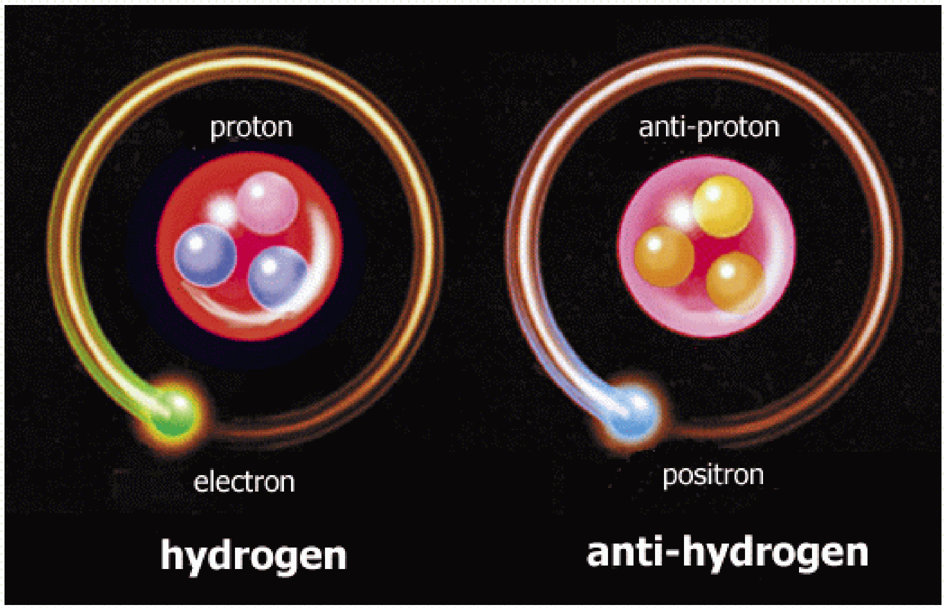
$$e^{+} + e^{-} \mapsto 2\gamma; 125\text{ns}$$

$$\mapsto 3\gamma; 142\text{ns}$$

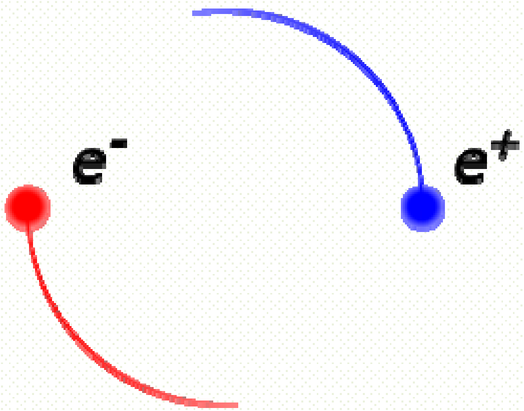


$$p^{+} + p^{-} \mapsto \gamma, e^{+}, e^{-}, \nu$$

$$e^{+} + p^{-} \mapsto ?1000\text{s}$$

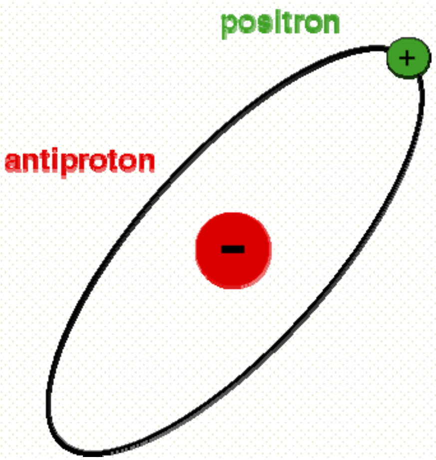


Anti-Matter



Positronium

Anti-夫妻



Antihydrogen

Antihelium

Anti-couple

Complex field vs. Real field

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x_i^{\ 2}} + V\psi = i\hbar\frac{\partial\psi}{\partial t}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$i\gamma \cdot \partial\psi = m\psi$$

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}; \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$\psi = \psi^* \quad ?$$

$$\gamma^0 = \beta$$

$$\gamma^k = \beta\alpha^k \qquad k = 1,2,3$$

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$$

Complex field vs. Real field

$\psi = \psi^*$?

Particles are their own antiparticles.

It was said that Photon (spin 1) and π^0 (spin zero) are their own antiparticles.

Antimatter, from Klein-Gordon equation?

Fermions, including charged electron and proton, and neutral neutron, are not their own antiparticles.

There are several categories of scientists in the world; those of second or third rank do their best but never get very far. Then there is the first rank, those who make important discoveries, fundamental to scientific progress. But then there are the geniuses, like Galilei and Newton. Majorana was one of these.

—(Enrico Fermi about Majorana, Rome 1938)

He disappeared suddenly under mysterious circumstances while going by ship from Palermo to Naples.

Ettore Majorana

1906-1938

In 1937, Majorana found that with the following matrix

$$\tilde{\gamma}^0 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \quad \tilde{\gamma}^1 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \quad \tilde{\gamma}^2 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \quad \tilde{\gamma}^3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{i}\tilde{\gamma} \cdot \partial \tilde{\psi} = \mathrm{m}\tilde{\psi} \qquad \tilde{\psi}^* = \psi$$

Majorana Fermion, but are they any?

E. Majorana, Nuovo Cimento 5, 171-184(1937)

Searching Majorana Fermion

Majorana speculated that his equation applies to neutrino, but neutrino (found in 1956) is itself speculative in 1937.

Lepton number conservation, and neutrinos oscillate in flavor, these facts favor distinction between neutrino and antineutrino.

$\nu_\mu + \mathrm{n} \to \mathrm{p} + \mu^-$ Yes! $\tilde{\nu}_\mu + \mathrm{n} \to \mathrm{p} + \mu^-$ No!

$\nu_\mu + \mathrm{p} \to \mathrm{n} + \mu^+$ No! $\tilde{\nu}_\mu + \mathrm{p} \to \mathrm{n} + \mu^+$ Yes!

United field theory: neutrinos be Majorana Fermion.

Searching Majorana Fermion

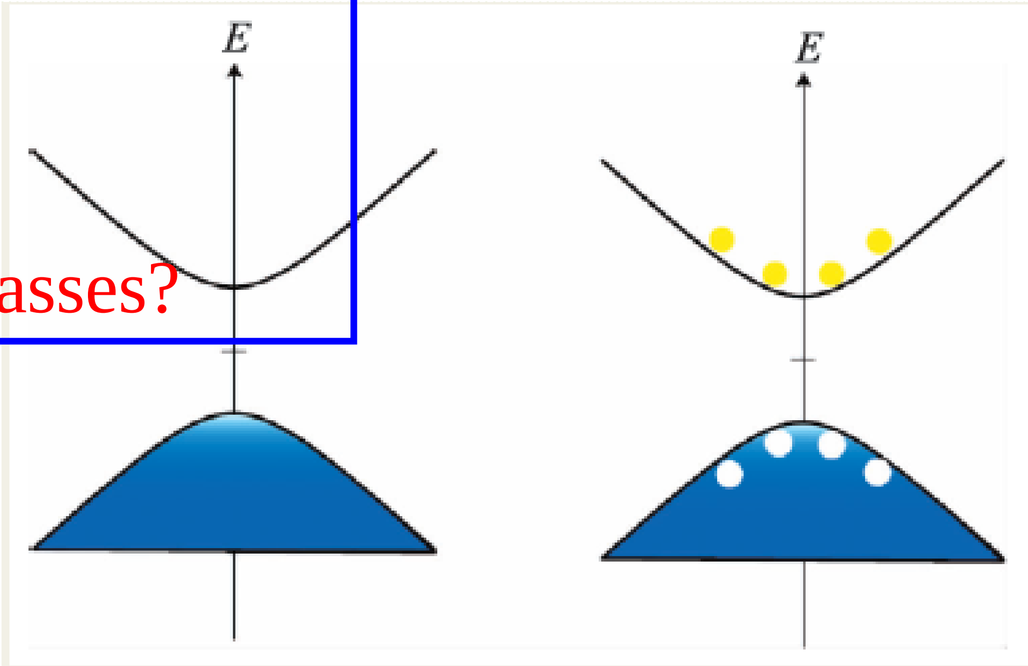
Majorana modes in Solid

Quasiparticle excitations in condensed-matter systems are Majorana fermions.

Mathematical Fantasy-Wilczek

Holes 'look' and 'behave' like the antiparticles or antimatter to Valence Electrons?

What about their effective masses?



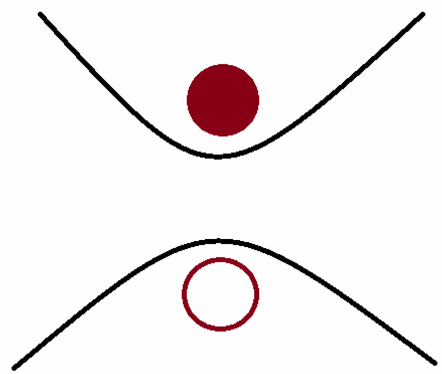
Searching Majorana Fermion

Particle–hole interchange (charge conjugation)

$$c_j \Leftrightarrow c_j^*$$

Excitons are bound states of electrons and holes, in the language of second quantization, are created by combined electron and hole operators

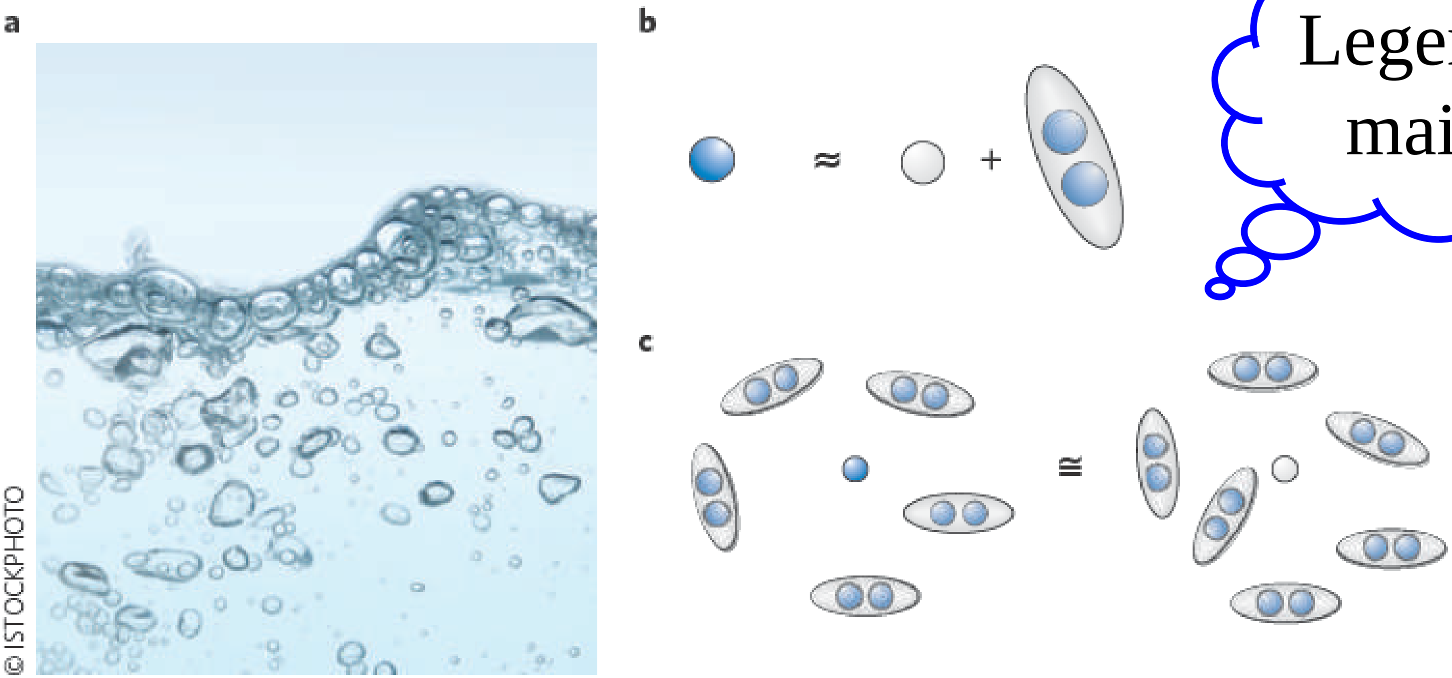
$$c_j^* c_k + c_k c_j^*$$



But conventional excitons are always bosons.

Searching Majorana Fermion

In superconductors the absolute distinction between electrons and holes is blurred



Legerde main

According to Bogliubov-Valatin formulism, creation operators for modes in superconducting state, which are their own particle at (j=k, π /4)

$$\cos\theta\, c_j + \sin\theta\, c_k^*$$

Searching Majorana Fermion

Majorana-type excitations emerge in certain types of superconductor, e.g., with Abrikosov vortices, the presence of which alters the equations for the electrons…… the vortices may trap so-called zero modes, spin-1/2 'excitons' of very low (~zero) energy.

The zero modes are discrete; there are a finite number associated with each vortex. The existence of these modes is related to the Atiyah–Singer index theorem **(The analytical index of an elliptical differential operator is equal to its topological index),** which connects the existence of special, symmetric solutions of differential equations to the **topology** of the parameters (E. J. Weinberg, *Phys. Rev*. D **24,** 2669–2673 (1981)).

Searching Majorana Fermion

Zero modes 'partiholes' are Majorana modes

$$\gamma = c_j + c_j^*$$

spin-1/2, symmetric under charge conjugation

$$c_j \Leftrightarrow c_j^*$$

Zero mode may occur if the Cooper pairs have orbital angular momentum 1 (p_x + ip_y-wave) (Quantum Hall states), or for *s*-wave Cooper pairing if the electrons in the normal state obey a Dirac-like equation (Topological Insulator).

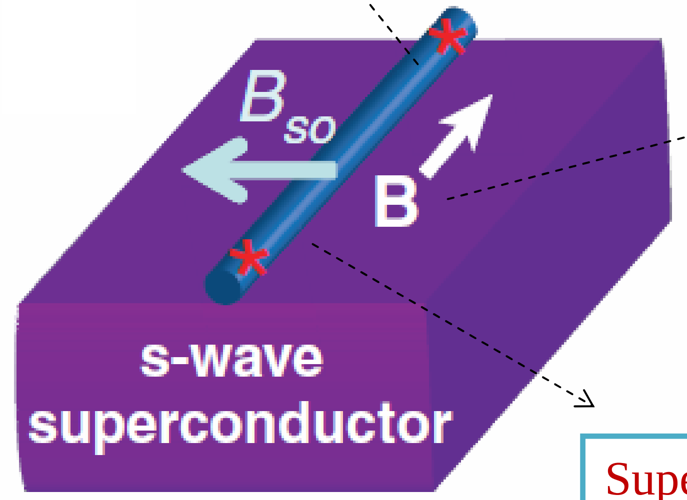
Majorana Recipe

In 2001, Alexei Kitaev predicted Majorana fermion to appear at each end of a superconducting wire.

- **One-dimentional quantum wire**
- **Spin-orbit interaction**
- **Superconductivity**
- **Magnetic field**

Roman M. Lutchyn *et al. Phys. Rev. Lett.* **105**, 077001 (2010); Yuval Oreg *et al. Phys. Rev. Lett.* **105**, 177002 (2010)

InSb nanowires (with strong spin-orbit interaction)



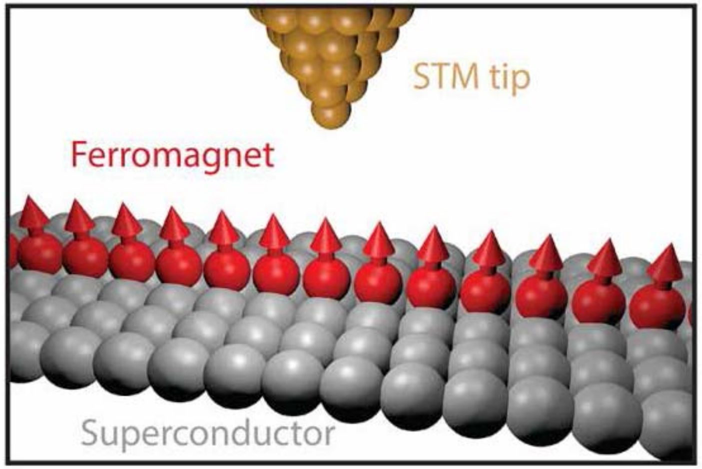
Magnetic field

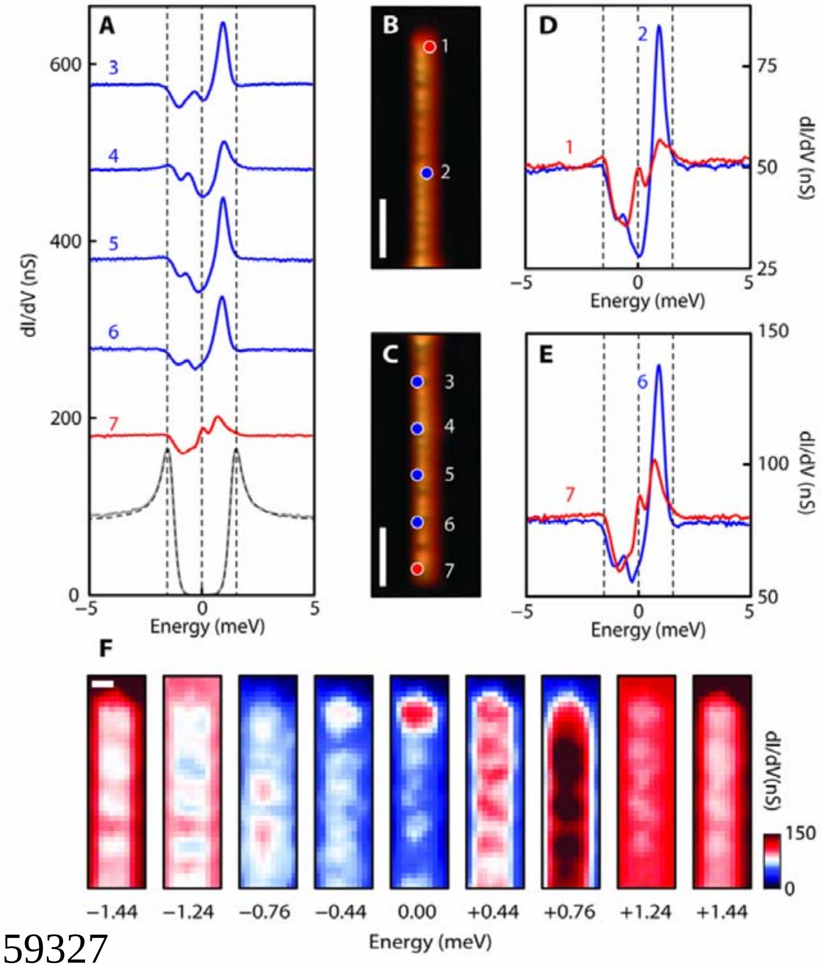
V. Mourik *et al. Science* **336,** 1003 (2012)

Superconducting proximity effects

Searching Majorana Fermion

- ➢**Fe chain**
- ➢**Ferromagnetic interaction between Fe atoms**
- ➢**strong spin-orbit interaction in superconducting Pb**





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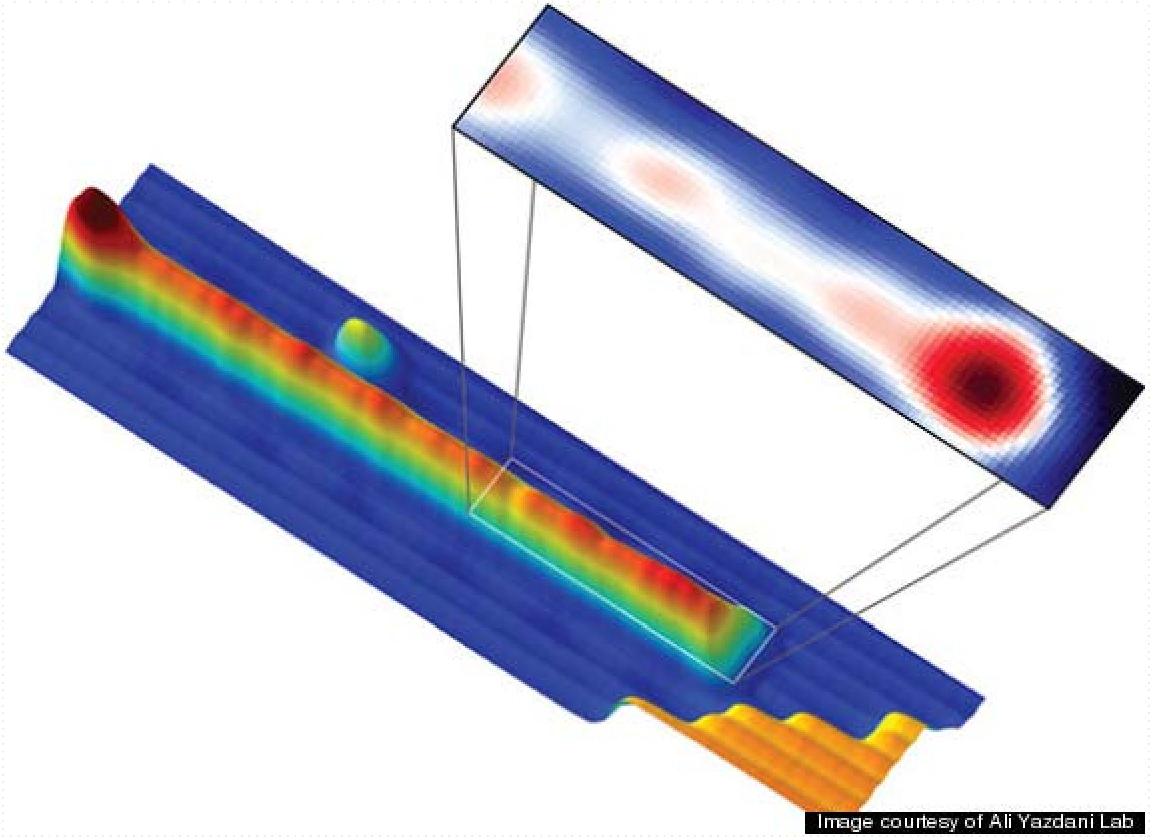
Searching Majorana Fermion



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Searching Majorana Fermion

Majorana fermions are predicted to localize at the edge of a topological superconductor, a state of matter that can form when a ferromagnetic system is placed in proximity to a conventional superconductor with strong spin-orbit interaction. With the goal of realizing a one-dimensional topological superconductor, we have fabricated ferromagnetic iron (Fe) atomic chains on the surface of superconducting lead (Pb). Using high-resolution spectroscopic imaging techniques, we show that the onset of superconductivity, which gaps the electronic density of states in the bulk of the Fe chains, is accompanied by the appearance of zero energy end states. This spatially resolved signature provides <u>strong evidence</u>, corroborated by other observations, for the formation of a topological phase and edge-bound Majorana fermions in our atomic chains.

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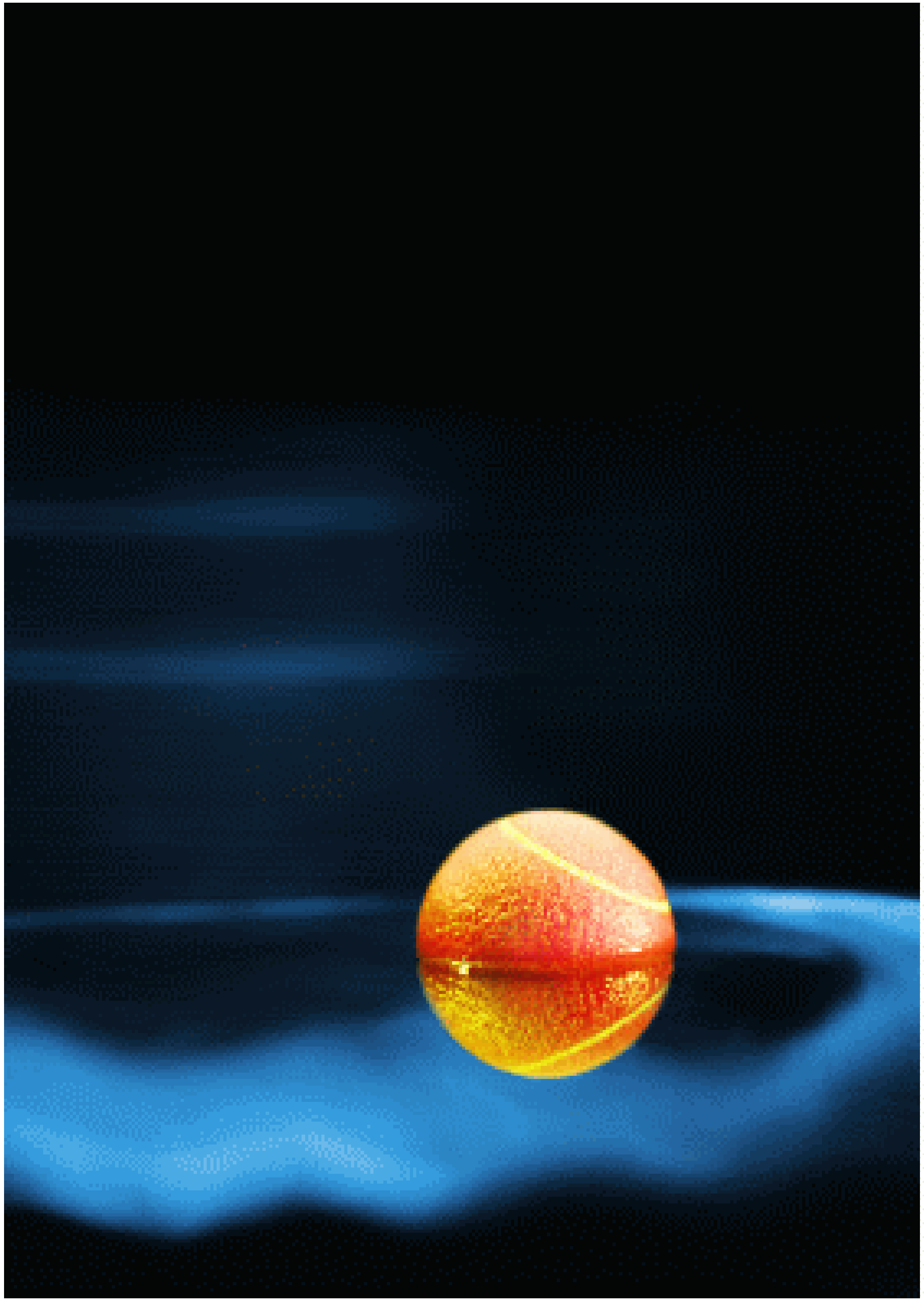

The Majorana fermion floating at the surface of the Fermi sea

Concluding remarks

- The concept of Majorana fermion arises from playing with Dirac Matrices;
- Majorana fermion in solids (modes, bound states) is a long story with too many Legerdemains;
- Some interesting STM patterns were obtained, which are referred to Majorana Fermion.

Schrödinger: Quantum mechanics was born in statistics and it will end in statistics

Statistics is referred to identical entities. Condensed matter systems are often complicated. One can directly look at a particle that obeys a clear statistics?

这只是一段粗浅的信口开河，**yet I wish it may be of some help to you！**